

Limit Distributions of Eigenvalues for Random Block Toeplitz and Hankel Matrices

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Abstract

Block Toeplitz and Hankel matrices arise in many aspects of applications. In this paper, we will research the distributions of eigenvalues for some models and get the semicircle law. Firstly we will give trace formulae of block Toeplitz and Hankel matrix. Then we will prove that the almost sure limit $\gamma_T^{(m)}$ ($\gamma_H^{(m)}$) of eigenvalue distributions of random block Toeplitz (Hankel) matrices exist and give the moments of the limit distributions where m is the order of the blocks. Then we will prove the existence of almost sure limit of eigenvalue distributions of random block Toeplitz and Hankel band matrices and give the moments of the limit distributions. Finally we will prove that $\gamma_T^{(m)}$ ($\gamma_H^{(m)}$) converges weakly to the semicircle law as $m \rightarrow \infty$.

Key words: Random block Toeplitz matrix; Hankel matrix; Eigenvalues distribution; Band matrix; Semicircle law.

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1 Introduction

In random matrix theory, a very important object is the eigenvalue distribution of a random matrix. If $A = (a_{ij}(\omega))_{i,j=1}^N$ is a real symmetric random matrix where the $(a_{ij}(\omega))$'s are random variables on a probability space Ω with a probability measure P , then the eigenvalue distribution of A is

$$\mu_A = \frac{1}{N} \int_{\Omega} \sum_{j=1}^N \delta_{\lambda_j(\omega)} dP(\omega)$$

where $\lambda_j(\omega)$'s are the N real eigenvalues of A .

The asymptotic behavior of the eigenvalue distribution is of much importance. In [23,24], Wigner got the semicircle law for a wide class of real symmetric random matrices and this great result caused much development of random matrix theory. Recently in a review paper (see [1]), Bai proposed the study of random matrix models with certain additional linear structure. The properties of the distributions of eigenvalues for random Hankel and Toeplitz matrices with independent entries are listed among the unsolved random matrix problems posed in [1]. In [5], Bryc, Dembo and Jiang proved the existence of limit distribution γ_T and γ_H of real symmetric Toeplitz and Hankel matrices. The moments of γ_T and γ_H are the sum of volumes of solids. Hammond and Miller in [9] also proved the existence of γ_T and γ_H independently. In [15], Liu and Wang proved the existence of limit distribution γ_T and γ_H for real symmetric, complex Hermitian band Toeplitz and real symmetric band Hankel matrices. We notice that Basak and Bose (see [2]) and Kargin (see [13]) also did the same work independently. Especially, the limit distribution of random Toeplitz band matrices with bandwidth $b_N = o(N)$ is Gaussian.

Block Toeplitz and Hankel matrices arise in many aspects of mathematics, physics and technology (see [8,10,14,19]). A block Toeplitz matrix is a block matrix which can be written as

$$T = (A_{i-j})_{i,j=1}^N = \begin{pmatrix} A_0 & A_{-1} & A_{-2} & \cdots & A_{-(N-1)} \\ A_1 & A_0 & A_{-1} & \cdots & A_{-(N-2)} \\ A_2 & A_1 & A_0 & \cdots & A_{-(N-3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{N-1} & A_{N-2} & A_{N-3} & \cdots & A_0 \end{pmatrix}$$

where $A_s = (a_{ij}(s))_{i,j=1}^m$ is an $m \times m$ matrix, $\forall s \in \{-N+1, \dots, N-1\}$. In [7], Gazzah, Regalia and Delmas researched the asymptotic behavior of the eigenvalue distribution for block Toeplitz matrices. In [18], Rashidi Far, Oraby, Bryc and Speicher have proved the existence of the limit distribution of eigenvalues for a random block Toeplitz matrix whose blocks are selfadjoint $m \times m$ matrices as $m \rightarrow \infty$, which implies that the double limit $\lim_{N \rightarrow \infty} \lim_{m \rightarrow \infty}$ gives the semicircle law. In this paper, we will study the limit distributions of real symmetric random block Toeplitz and Hankel matrices as $\lim_{N \rightarrow \infty}$ and $\lim_{m \rightarrow \infty} \lim_{N \rightarrow \infty}$.

In a block Toeplitz matrix with the form mentioned above, we suppose the $a_{ij}(s)$'s are real random variables. For symmetry, we need $A_s = (A_{-s})^T$. For independence of the elements, we suppose:

- (1) $a_{i_1 j_1}(s_1)$ and $a_{i_2 j_2}(s_2)$ are independent if $|s_1| \neq |s_2|$,
- (2) If $s \neq 0$ and $(i_1, j_1) \neq (i_2, j_2)$ then $a_{i_1 j_1}(s)$ and $a_{i_2 j_2}(s)$ are independent,
- (3) If $(i_1, j_1) \neq (i_2, j_2)$ and $(i_1, j_1) \neq (j_2, i_2)$ then $a_{i_1 j_1}(0)$ and $a_{i_2 j_2}(0)$ are independent.

In addition, we need the following uniform boundedness conditoin:

(4)

$$E(a_{ij}(s)) = 0, E(|a_{ij}(s)|^2) = 1, -(N-1) \leq s \leq N-1, 1 \leq i, j \leq m \quad (1.1)$$

and

$$\sup_{\substack{N \in \mathbb{N} \\ -(N-1) \leq s \leq N-1}} \{|a_{ij}(s)|^k | 1 \leq i, j \leq m\} = C_{k,m} < +\infty \quad (1.2)$$

and

$$\sup_{m \in \mathbb{N}} C_{k,m} = C_k < +\infty. \quad (1.3)$$

A block Hankel matrix is a block matrix which can be written as

$$H = (A_{N+1-i-j})_{i,j=1}^N = \begin{pmatrix} A_{N-1} & A_{N-2} & A_{N-3} & \cdots & A_0 \\ A_{N-2} & A_{N-3} & A_{N-4} & \cdots & A_{-1} \\ A_{N-3} & A_{N-4} & A_{N-5} & \cdots & A_{-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_0 & A_{-1} & A_{-2} & \cdots & A_{-(N-1)} \end{pmatrix}$$

where $A_s = (a_{ij}(s))_{i,j=1}^m$ is an $m \times m$ matrix, $\forall s \in \{-N+1, \dots, N-1\}$. Similar to random block Toeplitz matrices, we suppose the $a_{ij}(s)$'s are real random variables, $A_s = (A_s)^T$. In addition, we assume:

- (1) $a_{i_1 j_1}(s_1)$ and $a_{i_2 j_2}(s_2)$ are independent if $s_1 \neq s_2$,
- (2) If $(i_1, j_1) \neq (i_2, j_2)$ and $(i_1, j_1) \neq (j_2, i_2)$ then $a_{i_1 j_1}(s)$ and $a_{i_2 j_2}(s)$ are independent,
- (3)

$$E(a_{ij}(s)) = 0, E(|a_{ij}(s)|^2) = 1, -(N-1) \leq s \leq N-1, 1 \leq i, j \leq m$$

and

$$\sup_{\substack{N \in \mathbb{N} \\ -(N-1) \leq s \leq N-1}} \{|a_{ij}(s)|^k | 1 \leq i, j \leq m\} = C_{k,m} < +\infty$$

and

$$\sup_{m \in \mathbb{N}} C_{k,m} = C_k < +\infty.$$

We will firstly give trace formulae of block Toeplitz and Hankel matrices in Section 2. Using those trace formulae, we will prove that the almost sure limit distributions of block Toeplitz and Hankel matrices exist in Section 3. In that section we will also give the moments of the limit distributions. In Section 4, we will prove the existence of almost sure limit distributions of block Toeplitz and Hankel band matrices and give the moments of the limit distributions. In Section 5 we will prove that for block Toeplitz and Hankel matrices, the double limit $\lim_{m \rightarrow \infty} \lim_{N \rightarrow \infty}$ gives the semicircle law.

2 Trace Formulae of Block Toeplitz and Hankel Matrices

Definition 2.1. Let T be an $mN \times mN$ matrix and consist of N^2 blocks. If T has the form

$$T = (A_{i,j})_{i,j=1}^N = \begin{pmatrix} A_0 & A_{-1} & A_{-2} & \cdots & A_{-(N-1)} \\ A_1 & A_0 & A_{-1} & \cdots & A_{-(N-2)} \\ A_2 & A_1 & A_0 & \cdots & A_{-(N-3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{N-1} & A_{N-2} & A_{N-3} & \cdots & A_0 \end{pmatrix}$$

where $\{A_{-(N-1)}, \dots, A_0, \dots, A_{N-1}\}$ is a set of $m \times m$ matrices and $A_{i,j} = A_{i-j} = (a_{pq}(i-j))_{p,q=1}^m$, then we call T a block Toeplitz matrix.

Let H be an $mN \times mN$ matrix and can be written as $H = \Phi T$ where T is a block Toeplitz matrix and

$$\Phi = \begin{pmatrix} 0 & \cdots & 0 & I_m \\ 0 & \cdots & I_m & 0 \\ \vdots & \ddots & \vdots & \vdots \\ I_m & \cdots & 0 & 0 \end{pmatrix}$$

where I_m is the $m \times m$ unit matrix. Then we call H a block Hankel matrix.

For convenience, let $b_N = N - 1$.

Lemma 2.2. Let $T = (A_{i-j})_{i,j=1}^N$ be a block Toeplitz matrix and $A_s = (a_{pq}(s))_{p,q=1}^m$ where $-b_N \leq s \leq b_N$; $1 \leq p, q \leq m$. Then we have a trace

formula

$$\begin{aligned}\text{tr}(T^k) &= \sum_{i=1}^N \sum_{j_1, \dots, j_k = -b_N}^{b_N} \text{tr}(A_{j_1} \cdots A_{j_k}) \prod_{l=1}^k I_{[1, N]}(i + \sum_{q=1}^l j_q) \delta_{0, \sum_{q=1}^k j_q} \\ &= \sum_{i=1}^N \sum_{j_1, \dots, j_k = -b_N}^{b_N} \sum_{t_1, \dots, t_k = 1}^m a_{t_1 t_2}(j_1) \cdots a_{t_k t_1}(j_k) \prod_{l=1}^k I_{[1, N]}(i + \sum_{q=1}^l j_q) \delta_{0, \sum_{q=1}^k j_q}.\end{aligned}$$

Lemma 2.3. *Let $H = \Phi T$ be a block Hankel matrix and $T = (A_{i-j})_{i,j=1}^N$; $A_s = (a_{pq}(s))_{p,q=1}^m$ where $-b_N \leq s \leq b_N$; $1 \leq p, q \leq m$. Then we have the trace formula*

$$\begin{aligned}\text{tr}(H^k) &= \\ &\begin{cases} \sum_{i=1}^N \sum_{j_1, \dots, j_k = -b_N}^{b_N} \text{tr}(A_{j_1} \cdots A_{j_k}) \prod_{l=1}^k I_{[1, N]}(i - \sum_{q=1}^l (-1)^q j_q) \delta_{0, \sum_{q=1}^k (-1)^q j_q} & k \text{ even} \\ \sum_{i=1}^N \sum_{j_1, \dots, j_k = -b_N}^{b_N} \text{tr}(A_{j_1} \cdots A_{j_k}) \prod_{l=1}^k I_{[1, N]}(i - \sum_{q=1}^l (-1)^q j_q) \delta_{2i-1-N, \sum_{q=1}^k (-1)^q j_q} & k \text{ odd} \end{cases}\end{aligned}$$

where

$$\text{tr}(A_{j_1} \cdots A_{j_k}) = \sum_{t_1, \dots, t_k = 1}^m a_{t_1 t_2}(j_1) \cdots a_{t_k t_1}(j_k).$$

To prove the above lemmas, we consider Kronecker product of two matrices (see [12]). Let $A = (a_{ij})$ be an $m \times n$ matrix and B be a $p \times q$ matrix. The Kronecker product of A and B is an $mp \times nq$ matrix:

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}.$$

Proof of Lemma 2.2. Let B and F be two $N \times N$ matrices and

$$B = (\delta_{i+1,j})_{i,j=1}^N = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, F = (\delta_{i,j+1})_{i,j=1}^N = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Then $T = \sum_{j=0}^{b_N} B^j \otimes A_{-j} + \sum_{j=1}^{b_N} F^j \otimes A_j$. Let e_1^i be the i th unit vector in \mathbb{R}^N and e_2^j be the j th unit vector in \mathbb{R}^m , then we have

$$B^l \otimes A_{-l}(e_1^i \otimes e_2^j) = (B^l e_1^i) \otimes (A_{-l} e_2^j) = I_{[1,N]}(i-l)e_1^{i-l} \otimes (A_{-l} e_2^j)$$

and

$$F^l \otimes A_l(e_1^i \otimes e_2^j) = (F^l e_1^i) \otimes (A_l e_2^j) = I_{[1,N]}(i+l)e_1^{i+l} \otimes (A_l e_2^j)$$

thus

$$\begin{aligned} T(e_1^i \otimes e_2^j) &= \sum_{l=0}^{b_N} I_{[1,N]}(i-l)e_1^{i-l} \otimes (A_{-l} e_2^j) + \sum_{l=1}^{b_N} I_{[1,N]}(i+l)e_1^{i+l} \otimes (A_l e_2^j) \\ &= \sum_{l=-b_N}^{b_N} I_{[1,N]}(i+l)e_1^{i+l} \otimes (A_l e_2^j), \end{aligned}$$

so

$$T^k(e_1^i \otimes e_2^j) = \sum_{l_1, \dots, l_k = -b_N}^{b_N} \prod_{r=1}^k I_{[1,N]}(i + \sum_{q=1}^r l_q) e_1^{i + \sum_{q=1}^r l_q} \otimes (A_{l_k} \cdots A_{l_1} e_2^j).$$

As $\{e_1^i \otimes e_2^j \mid 1 \leq i \leq N; 1 \leq j \leq m\}$ is a standard basis of \mathbb{R}^{mN} , we have the trace formula

$$\begin{aligned} &\text{tr}(T^k) \\ &= \sum_{i=1}^N \sum_{j=1}^m (e_1^i \otimes e_2^j)^T T^k(e_1^i \otimes e_2^j) \\ &= \sum_{i=1}^N \sum_{l_1, \dots, l_k = -b_N}^{b_N} \prod_{r=1}^k I_{[1,N]}(i + \sum_{q=1}^r l_q) \sum_{j=1}^m ((e_1^i)^T \otimes (e_2^j)^T) \cdot (e_1^{i + \sum_{q=1}^r l_q} \otimes (A_{l_k} \cdots A_{l_1} e_2^j)) \\ &= \sum_{i=1}^N \sum_{l_1, \dots, l_k = -b_N}^{b_N} \prod_{r=1}^k I_{[1,N]}(i + \sum_{q=1}^r l_q) \delta_{0, \sum_{q=1}^k l_q} \cdot \sum_{j=1}^m ((e_2^j)^T A_{l_k} \cdots A_{l_1} e_2^j) \\ &= \sum_{i=1}^N \sum_{l_1, \dots, l_k = -b_N}^{b_N} \prod_{r=1}^k I_{[1,N]}(i + \sum_{q=1}^r l_q) \text{tr}(A_{l_k} \cdots A_{l_1}) \delta_{0, \sum_{q=1}^k l_q}. \end{aligned}$$

For $\text{tr}(A_{l_k} \cdots A_{l_1}) = \text{tr}(A_{l_1} \cdots A_{l_k})$, we get

$$\text{tr}(T^k) = \sum_{i=1}^N \sum_{j_1, \dots, j_k = -b_N}^{b_N} \text{tr}(A_{j_1} \cdots A_{j_k}) \prod_{l=1}^k I_{[1, N]}(i + \sum_{q=1}^l j_q) \delta_{0, \sum_{q=1}^k j_q}.$$

Note that

$$\text{tr}(A_{l_1} \cdots A_{l_k}) = \sum_{t_1, \dots, t_k=1}^m a_{t_1 t_2}(l_1) \cdots a_{t_k t_1}(l_k),$$

and we directly get

$$\text{tr}(T^k) = \sum_{i=1}^N \sum_{j_1, \dots, j_k = -b_N}^{b_N} \sum_{t_1, \dots, t_k=1}^m a_{t_1 t_2}(j_1) \cdots a_{t_k t_1}(j_k) \prod_{l=1}^k I_{[1, N]}(i + \sum_{q=1}^l j_q) \delta_{0, \sum_{q=1}^k j_q}.$$

□

Proof of Lemma 2.3. $\Phi = P \otimes I_m$ where I_m is the $m \times m$ unit matrix and P is an $N \times N$ matrix and

$$P = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{pmatrix}.$$

So if A is an $N \times N$ matrix and B is an $m \times m$ matrix, then we have $\Phi(A \otimes B) = (P \otimes I_m)(A \otimes B) = (PA) \otimes B$. Note that $P e_1^i = e_1^{N+1-i}$.

As in the proof of Lemma 2.2, we have

$$\Phi T(e_1^i \otimes e_2^j) = (P \otimes I_m) \left(\sum_{l=-b_N}^{b_N} I_{[1, N]}(i+l) e_1^{i+l} \otimes (A^l e_2^j) \right) = \sum_{l=-b_N}^{b_N} I_{[1, N]}(i+l) e_1^{N+1-(i+l)} \otimes (A^l e_2^j),$$

and then

$$H^k(e_1^i \otimes e_2^j) = \begin{cases} \sum_{j_1, \dots, j_k = -b_N}^{b_N} \prod_{r=1}^k I_{[1, N]}(i - \sum_{q=1}^r (-1)^q l_q) e_1^{i - \sum_{q=1}^k (-1)^q j_q} \otimes (A_{l_k} \cdots A_{l_1}) e_2^j & k \text{ even,} \\ \sum_{j_1, \dots, j_k = -b_N}^{b_N} \prod_{r=1}^k I_{[1, N]}(i - \sum_{q=1}^r (-1)^q l_q) e_1^{N+1-(i - \sum_{q=1}^k (-1)^q j_q)} \otimes (A_{l_k} \cdots A_{l_1}) e_2^j & k \text{ odd.} \end{cases}$$

Thus we get

$$\begin{aligned}
& \text{tr}(H^k) \\
&= \sum_{i=1}^N \sum_{j=1}^m (e_1^i \otimes e_2^j)^T H^k (e_1^i \otimes e_2^j) \\
&= \begin{cases} \sum_{i=1}^N \sum_{j_1, \dots, j_k = -b_N}^{b_N} \text{tr}(A_{j_1} \cdots A_{j_k}) \prod_{l=1}^k I_{[1, N]}(i - \sum_{q=1}^l (-1)^q j_q) \delta_{0, \sum_{q=1}^k (-1)^q j_q} & k \text{ even} \\ \sum_{i=1}^N \sum_{j_1, \dots, j_k = -b_N}^{b_N} \text{tr}(A_{j_1} \cdots A_{j_k}) \prod_{l=1}^k I_{[1, N]}(i - \sum_{q=1}^l (-1)^q j_q) \delta_{2i-1-N, \sum_{q=1}^k (-1)^q j_q} & k \text{ odd} \end{cases}
\end{aligned}$$

where

$$\text{tr}(A_{j_1} \cdots A_{j_k}) = \sum_{t_1, \dots, t_k=1}^m a_{t_1 t_2}(j_1) \cdots a_{t_k t_1}(j_k).$$

□

3 Limit Distributions of Random Block Toeplitz and Hankel Matrices

Let $\#V$ denote the number of elements in an arbitrary finite set V . We now review the concept of partition (see [15]). Let $[n] = \{1, 2, \dots, n\}$.

- (1) We call $\pi = \{V_1, V_2, \dots, V_r\}$ a partition of $[n]$ if $\bigcup_{j=1}^r V_j = [n]$ and $V_i \cap V_j = \emptyset$ if $i \neq j$.
- (2) For $\forall p \in [n]$, we define $\pi(p) = i$ if $p \in V_i$. We write $p \sim_\pi q$ if $\pi(p) = \pi(q)$.
- (3) Let $\mathcal{P}(n)$ denote the set of all the partition of $[n]$. We define $\mathcal{P}_2(n) = \{\pi = \{V_1, V_2, \dots, V_r\} \in \mathcal{P}(n) \mid \#V_i = 2, \forall i\}$ and $\mathcal{P}_2^1(n) = \{\pi = \{V_1, V_2, \dots, V_r\} \in \mathcal{P}_2(n) \mid V_i \text{ has exactly one even element and one odd element, } \forall i\}$.

Definition 3.1. Let $k \in \mathbb{N}$, $\pi = \{V_1, V_2, \dots, V_k\} \in \mathcal{P}_2(2k)$, $V_r = \{a_r, b_r\}$ ($1 \leq r \leq k$), then π determines a system of linear equations with unknown variables t_1, t_2, \dots, t_{2k} (set $t_{2k+1} = t_1$):

$$\begin{cases} t_{a_i} = t_{b_{i+1}} & 1 \leq i \leq k \\ t_{b_i} = t_{a_{i+1}} & 1 \leq i \leq k \end{cases}. \quad (3.1)$$

The number of linearly independent equations of this system is denoted by $f(\pi)$. Clearly we have $0 \leq f(\pi) \leq 2k$.

Theorem 3.2. Let $T_N = (A_{i-j})_{i,j=1}^N$ be an $mN \times mN$ random block Toeplitz matrix where $A_s = (a_{ij}(s))_{i,j=1}^m$, $A_{-s} = (A_s)^T$, and $a_{ij}(s)$'s are random variables satisfying the four conditions mentioned in introduction. Let $X_N = \frac{T_N}{\sqrt{mN}}$, then eigenvalue distribution μ_{X_N} converges almost surely to a symmetric probability distribution $\gamma_T^{(m)}$ which is determined by its even moments

$$m_{2k}(\gamma_T^{(m)}) = \sum_{\pi \in \mathcal{P}_2(2k)} m^{k-1-f(\pi)} \int_{[0,1] \times [-1,1]^k} \prod_{j=1}^{2k} I_{[0,1]}(x_0 + \sum_{q=1}^j \epsilon_\pi(q) x_{\pi(q)}) \prod_{l=0}^k dx_l$$

where

$$\epsilon_\pi(q) = \begin{cases} 1 & \text{if } q \text{ is the smaller element of } V_{\pi(q)}, \\ -1 & \text{if } q \text{ is the larger element of } V_{\pi(q)}. \end{cases} \quad (3.2)$$

Proof of Theorem 3.2. The k th moment of μ_{X_N} is given by

$$m_{k,N} = \int x^k d\mu_{X_N} = \frac{1}{mN} E(\text{tr} X_N^k) = \frac{1}{mN} (mN)^{-\frac{k}{2}} E(\text{tr} T_N^k).$$

Using Lemma 2.2, we get

$$m_{k,N} = \frac{1}{mN} (mN)^{-\frac{k}{2}} \sum_{i=1}^N \sum_{j_1, \dots, j_k = -b_N}^{b_N} \sum_{t_1, \dots, t_k = 1}^m E(a_{t_1 t_2}(j_1) \cdots a_{t_k t_1}(j_k)) \prod_{l=1}^k I_{[1,N]}(i + \sum_{q=1}^l j_q) \delta_{0, \sum_{q=1}^k j_q}.$$

We will calculate the limit $\lim_{N \rightarrow \infty} m_{k,N}$.

Observe that

$$|E(a_{t_1 t_2}(j_1) \cdots a_{t_k t_1}(j_k))| \leq D_k \quad (\forall j_1, \dots, j_k, t_1, \dots, t_k),$$

where $D_k = (\max\{C_1, \dots, C_k\})^k$. If $|j_p| \neq |j_q|$, then $a_{t_p t_{p+1}}(j_p)$ and $a_{t_q t_{q+1}}(j_q)$ are independent. Let $p \in \{1, 2, \dots, k\}$. By the independence conditions and $E(a_{ij}(s)) = 0$, we observe that if $E(a_{t_1 t_2}(j_1) \cdots a_{t_k t_1}(j_k)) \neq 0$, then for any $p \in \{1, \dots, k\}$, $\exists q \in \{1, \dots, k\}$, $q \neq p$, s.t. $|j_q| = |j_p|$. So there are at most $\lceil \frac{k}{2} \rceil$ different elements in $V = \{|j_1|, \dots, |j_k|\}$. Further, we have

$$\begin{aligned} \left| \sum_{j_1, \dots, j_k = -b_N}^{b_N} E(a_{t_1 t_2}(j_1) \cdots a_{t_k t_1}(j_k)) \right| &= \left| \sum_{s=1}^{\lceil \frac{k}{2} \rceil} \sum_{\#V=s} E(a_{t_1 t_2}(j_1) \cdots a_{t_k t_1}(j_k)) \right| \\ &\leq \sum_{s=1}^{\lceil \frac{k}{2} \rceil} \eta_{k,s} \frac{(b_N + 1)!}{(b_N + 1 - s)!} 2^s D_k \end{aligned}$$

where $\eta_{k,s} = \# \left\{ \pi = \{U_1, \dots, U_s\} \in \mathcal{P}(k) \mid \#U_i \geq 2, \forall i \right\}$.

So $\left| \sum_{j_1, \dots, j_k = -b_N}^{b_N} E(a_{t_1 t_2}(j_1) \cdots a_{t_k t_1}(j_k)) \right| = O(N^{\lfloor \frac{k}{2} \rfloor})$ and then $m_{k,N} = O(N^{\lfloor \frac{k}{2} \rfloor - \frac{k}{2}})$.

Thus for odd k , $m_{k,N} = o(1)$. Now we only have to consider $m_{2k,N}$.

π is a partition of $[2k] = \{1, 2, \dots, 2k\}$ and $p \sim_\pi q \iff |j_p| = |j_q|$, then we have

$$\begin{aligned} & m_{2k,N} \\ &= \sum_{i=1}^N \sum_{\pi \in \mathcal{P}(2k)} \sum_{\substack{j_1, \dots, j_{2k} = -b_N \\ p \sim_\pi q \iff |j_p| = |j_q|}}^{b_N} \sum_{t_1, \dots, t_{2k}=1}^m \frac{E(a_{t_1 t_2}(j_1) \cdots a_{t_{2k} t_1}(j_{2k}))}{(mN)^{k+1}} \prod_{l=1}^{2k} I_{[1,N]}(i + \sum_{q=1}^l j_q) \delta_{0, \sum_{q=1}^{2k} j_q} \\ &+ o(1). \end{aligned} \tag{3.3}$$

Again by the assumptions in the introduction, the contribution of the partitions which are not pair partition to $m_{2k,N}$ is $o(1)$. So we only have to consider the pair partitions. Suppose $\pi \in \mathcal{P}_2(2k)$. If $p \sim_\pi q$, then $j_p = j_q$ or $j_p = -j_q$. Under the condition $\sum_{q=1}^{2k} j_q = 0$ according to (3.3), considering the main contribution to the trace, we should take $j_p = -j_q$. Otherwise there exists $p_0, q_0 \in [2k]$ such that

$$j_{p_0} = j_{q_0} = \frac{1}{2}(j_{p_0} + j_{q_0} - \sum_{q=1}^{2k} j_q).$$

We can choose other $k-1$ distinct numbers, which determine $j_{p_0} = j_{q_0}$ and then there is a loss of at least one degree of freedom and the contribution of such terms is $O(N^{-1})$. Therefore we have

$$\begin{aligned} & m_{2k,N} \\ &= \sum_{i=1}^N \sum_{\pi \in \mathcal{P}_2(2k)} \sum_{\substack{j_1, \dots, j_{2k} = -b_N \\ p \sim_\pi q \iff j_p = -j_q}}^{b_N} \sum_{t_1, \dots, t_{2k}=1}^m \frac{E(a_{t_1 t_2}(j_1) \cdots a_{t_{2k} t_1}(j_{2k}))}{(mN)^{k+1}} \prod_{l=1}^{2k} I_{[1,N]}(i + \sum_{q=1}^l j_q) \\ &+ o(1) \\ &= \sum_{i=1}^N \sum_{\pi \in \mathcal{P}_2(2k)} \sum_{\substack{j_1, \dots, j_{2k} = -b_N \\ j_t \neq 0, \forall t \\ p \sim_\pi q \iff j_p = -j_q}}^{b_N} \sum_{t_1, \dots, t_{2k}=1}^m \frac{E(a_{t_1 t_2}(j_1) \cdots a_{t_{2k} t_1}(j_{2k}))}{(mN)^{k+1}} \prod_{l=1}^{2k} I_{[1,N]}(i + \sum_{q=1}^l j_q) \\ &+ o(1). \end{aligned}$$

Now we have to calculate

$$\sum_{t_1, \dots, t_{2k}=1}^m E(a_{t_1 t_2}(j_1) \cdots a_{t_{2k} t_1}(j_{2k}))$$

with the assumption that $j_t \neq 0, \forall t$. For convenience we let $t_{2k+1} = t_1$. Suppose

$$E(a_{t_1 t_2}(j_1) \cdots a_{t_{2k} t_1}(j_{2k})) \neq 0,$$

then for any $p \sim_\pi q$ we have $a_{t_p t_{p+1}}(j_p) = a_{t_q t_{q+1}}(j_q)$. Suppose $p < q$.

$$\begin{aligned} & E(a_{t_1 t_2}(j_1) \cdots a_{t_{2k} t_1}(j_{2k})) \\ &= E(a_{t_p t_{p+1}}(j_p) a_{t_q t_{q+1}}(j_q)) \cdot E(\widehat{a_{t_1 t_2}(j_1)} \cdots \widehat{a_{t_p t_{p+1}}(j_p)} \cdots \widehat{a_{t_q t_{q+1}}(j_q)} \cdots a_{t_{2k} t_1}(j_{2k})) \\ &= 1 \cdot E(a_{t_1 t_2}(j_1) \cdots \widehat{a_{t_p t_{p+1}}(j_p)} \cdots \widehat{a_{t_q t_{q+1}}(j_q)} \cdots a_{t_{2k} t_1}(j_{2k})) \end{aligned}$$

where $\widehat{}$ marks omitted index. Note that $j_p = -j_q$, $A_{j_p} = (A_{j_q})^T$ and $A_{j_p} \neq A_{j_q}$ because $j_p = -j_q \neq 0$. $a_{t_p t_{p+1}}(j_p)$ lies in the t_p -th row and the t_{p+1} -th column of A_{j_p} and $a_{t_q t_{q+1}}(j_q)$ lies in the t_q -th row and the t_{q+1} -th column of A_{j_q} . So $t_p = t_{q+1}$, $t_q = t_{p+1}$. As $\pi = \{V_1, V_2, \dots, V_r\}$ where $V_i = \{a_i, b_i\}$ ($1 \leq i \leq r$), we have a system of equations:

$$\begin{cases} t_{a_1} = t_{b_1+1} \\ t_{a_1+1} = t_{b_1} \\ \dots \\ t_{a_k} = t_{b_k+1} \\ t_{a_k+1} = t_{b_k} \end{cases}.$$

By Definition 3.1, there are $f(\pi)$ linearly independent equations in this system. So there are $2k - f(\pi)$ variables taking values freely in $\{1, 2, \dots, m\}$ and the number of solutions of this system of equations is $m^{2k-f(\pi)}$. In other words, there are $m^{2k-f(\pi)}$ different (t_1, \dots, t_{2k}) 's such that

$$E(a_{t_1 t_2}(j_1) \cdots a_{t_{2k} t_1}(j_{2k})) \neq 0$$

and this implies

$$E(a_{t_1 t_2}(j_1) \cdots a_{t_{2k} t_1}(j_{2k})) = 1.$$

So

$$\sum_{t_1, \dots, t_{2k}=1}^m E(a_{t_1 t_2}(j_1) \cdots a_{t_{2k} t_1}(j_{2k})) = m^{2k-f(\pi)}.$$

Thus we have

$$\begin{aligned}
& m_{2k,N} \\
&= \frac{1}{(mN)^{k+1}} \sum_{i=1}^N \sum_{\pi \in \mathcal{P}_2(2k)} m^{2k-f(\pi)} \sum_{\substack{j_1, \dots, j_{2k} = -b_N \\ j_t \neq 0, \forall t \\ p \sim_{\pi} q \iff j_p = -j_q}}^{b_N} \prod_{l=1}^{2k} I_{[1,N]}(i + \sum_{q=1}^l j_q) \\
&+ o(1) \\
&= \frac{1}{(mN)^{k+1}} \sum_{i=1}^N \sum_{\pi \in \mathcal{P}_2(2k)} m^{2k-f(\pi)} \sum_{\substack{j_1, \dots, j_{2k} = -b_N \\ p \sim_{\pi} q \iff j_p = -j_q}}^{b_N} \prod_{l=1}^{2k} I_{[1,N]}(i + \sum_{q=1}^l j_q) \\
&+ o(1).
\end{aligned}$$

Now for any $r \in \{1, \dots, k\}$, let $x_r = j_{a_r}$, then $j_q = \begin{cases} x_r & \text{if } q = a_r \\ -x_r & \text{if } q = b_r \end{cases}$.

Remember Eq.(3.2), and then we have $j_q = \begin{cases} x_{\pi(q)} & \text{if } \epsilon_{\pi}(q) = 1 \\ -x_{\pi(q)} & \text{if } \epsilon_{\pi}(q) = -1 \end{cases}$, so $j_q = \epsilon_{\pi}(q)x_{\pi(q)}$ and

$$m_{2k,N} = (mN)^{-k-1} \sum_{i=1}^N \sum_{\pi \in \mathcal{P}_2(2k)} \sum_{x_1, \dots, x_k = -b_N}^{b_N} m^{2k-f(\pi)} \prod_{l=1}^{2k} I_{[1,N]}(i + \sum_{q=1}^l \epsilon_{\pi}(q)x_{\pi(q)}) + o(1),$$

thus

$$\lim_{N \rightarrow \infty} m_{2k,N} = \sum_{\pi \in \mathcal{P}_2(2k)} m^{k-1-f(\pi)} \int_{[0,1] \times [-1,1]^k} \prod_{j=1}^{2k} I_{[0,1]}(x_0 + \sum_{q=1}^j \epsilon_{\pi}(q) x_{\pi(q)}) \prod_{l=0}^k dx_l.$$

$$\text{Let } m_{2k} = \sum_{\pi \in \mathcal{P}_2(2k)} m^{k-1-f(\pi)} \int_{[0,1] \times [-1,1]^k} \prod_{j=1}^{2k} I_{[0,1]}(x_0 + \sum_{q=1}^j \epsilon_{\pi}(q) x_{\pi(q)}) \prod_{l=0}^k dx_l$$

and $m_{2k-1} = 0 (k \in \mathbb{N})$, then for any $k \in \mathbb{N}$ we have $\lim_{N \rightarrow \infty} m_{k,N} = m_k$. It is easy to see that $m_{2k} \leq \sum_{\pi \in \mathcal{P}_2(2k)} m^{k-1-f(\pi)} \leq (2k-1)!! \cdot m^{k-1}$ and then

using Carleman's theorem (see [6]) we know that the limit distribution $\gamma_T^{(m)}$ is uniquely determined by its moments $\{m_k\}_{k=0}^{\infty}$.

Now we prove the almost sure convergence. It is sufficient to prove

$$\sum_{N=1}^{\infty} \frac{1}{N^4} E\left((\text{tr} X_N^k - E(\text{tr} X_N^k))^4\right) < \infty. \quad (3.4)$$

$$\begin{aligned}
\text{tr} X_N^k &= (mN)^{-\frac{k}{2}} \text{tr} T_N^k \\
&= (mN)^{-\frac{k}{2}} \sum_{i=1}^N \sum_{t_1, \dots, t_k=1}^m \sum_{j_1, \dots, j_k=-b_N}^{b_N} a_{t_1 t_2}(j_1) \cdots a_{t_k t_1}(j_k) \prod_{l=1}^k I_{[1, N]}(i + \sum_{q=1}^l j_q) \delta_{0, \sum_{q=1}^k j_q}.
\end{aligned}$$

For convenience we let

$$\begin{aligned}
\mathbf{j} &= (j_1, \dots, j_k), \quad \mathbf{t} = (t_1, \dots, t_k), \\
A[i, \mathbf{t}, \mathbf{j}] &= a_{t_1 t_2}(j_1) \cdots a_{t_k t_1}(j_k) \prod_{l=1}^k I_{[1, N]}(i + \sum_{q=1}^l j_q) \delta_{0, \sum_{q=1}^k j_q},
\end{aligned}$$

and we use $\sum_{i, \mathbf{t}, \mathbf{j}}$ to denote $\sum_{i=1}^N \sum_{t_1, \dots, t_k=1}^m \sum_{j_1, \dots, j_k=-b_N}^{b_N}$, then we have

$$\begin{aligned}
&\frac{1}{N^4} E \left((\text{tr} X_N^k - E(\text{tr} X_N^k))^4 \right) \\
&= \frac{1}{N^4} E \left((mN)^{-\frac{k}{2}} \sum_{i, \mathbf{t}, \mathbf{j}} A[i, \mathbf{t}, \mathbf{j}] - E((mN)^{-\frac{k}{2}} \sum_{i, \mathbf{t}, \mathbf{j}} A[i, \mathbf{t}, \mathbf{j}]) \right)^4 \\
&= \frac{1}{N^{2k+4} m^{2k}} E \left(\sum_{i, \mathbf{t}, \mathbf{j}} (A[i, \mathbf{t}, \mathbf{j}] - EA[i, \mathbf{t}, \mathbf{j}]) \right)^4 \\
&= \frac{1}{N^{2k+4} m^{2k}} E \left(\sum_{v=1}^4 \sum_{i^v, \mathbf{t}^v, \mathbf{j}^v} \prod_{v=1}^4 (A[i^v, \mathbf{t}^v, \mathbf{j}^v] - EA[i^v, \mathbf{t}^v, \mathbf{j}^v]) \right)
\end{aligned}$$

where $\sum_{i^v, \mathbf{t}^v, \mathbf{j}^v} = \sum_{i^v=1}^N \sum_{t_1^v, \dots, t_k^v=1}^m \sum_{j_1^v, \dots, j_k^v=-b_N}^{b_N}$.

For given $\mathbf{j}^1 = (j_1^1, \dots, j_k^1)$, $\mathbf{j}^2 = (j_1^2, \dots, j_k^2)$, $\mathbf{j}^3 = (j_1^3, \dots, j_k^3)$, $\mathbf{j}^4 = (j_1^4, \dots, j_k^4)$, let

$$\mathbf{J} = (j_1^1, \dots, j_k^1, \dots, j_1^4, \dots, j_k^4) \in \{-b_N, \dots, b_N\}^{4k}.$$

Set $S^{\mathbf{J}} = \{|j_1^1|, \dots, |j_k^1|, \dots, |j_1^4|, \dots, |j_k^4|\}$. We use $p(\mathbf{J})$ to denote the number of different elements of $S^{\mathbf{J}}$.

We construct a set of numbers with multiplicities $S_{\mathbf{J}} = \{j_1^1, \dots, j_k^1, \dots, j_1^4, \dots, j_k^4\}$.

Please note that if $|j_{u_1}^{v_1}| = |j_{u_2}^{v_2}| = a$, then a appears twice in $S_{\mathbf{J}}$. Let

$S_1, \dots, S_{p(\mathbf{J})}$ be subsets of $S_{\mathbf{J}}$ such that:

- (a) for all w , the elements of S_w have the same absolute value;
- (b) if $w_1 \neq w_2$, then the absolute value of the elements in S_{w_1} is different from the absolute value of the elements in S_{w_2} ;

(c) $\bigcup_{r=1}^p S_r = S_{\mathbf{J}}$. It is easy to see that $S_1, \dots, S_{p(\mathbf{J})}$ are uniquely determined by \mathbf{J} . Then we have

$$\{-b_N, \dots, b_N\}^{4k} = W_1 \cup W_2 \cup W_3$$

where

$$W_1 = \{\mathbf{J} \in \{-b_N, \dots, b_N\}^{4k} | p(\mathbf{J}) \leq 2k - 2\},$$

$$W_2 = \{\mathbf{J} \in \{-b_N, \dots, b_N\}^{4k} | p(\mathbf{J}) = 2k - 1\},$$

$$W_3 = \{\mathbf{J} \in \{-b_N, \dots, b_N\}^{4k} | p(\mathbf{J}) = 2k\}.$$

Therefore we have

$$\begin{aligned} & \frac{1}{N^4} E \left((\text{tr} X_N^k - E(\text{tr} X_N^k))^4 \right) \\ &= \frac{1}{N^{2k+4} m^{2k}} E \left(\sum_{\substack{i^1, i^2, i^3, i^4 \\ \mathbf{t}^1, \mathbf{t}^2, \mathbf{t}^3, \mathbf{t}^4}} \sum_{\mathbf{J} \in W_1} \prod_{v=1}^4 (A[i^v, \mathbf{t}^v, \mathbf{j}^v] - EA[i^v, \mathbf{t}^v, \mathbf{j}^v]) \right) \\ &+ \frac{1}{N^{2k+4} m^{2k}} E \left(\sum_{\substack{i^1, i^2, i^3, i^4 \\ \mathbf{t}^1, \mathbf{t}^2, \mathbf{t}^3, \mathbf{t}^4}} \sum_{\mathbf{J} \in W_2} \prod_{v=1}^4 (A[i^v, \mathbf{t}^v, \mathbf{j}^v] - EA[i^v, \mathbf{t}^v, \mathbf{j}^v]) \right) \\ &+ \frac{1}{N^{2k+4} m^{2k}} E \left(\sum_{\substack{i^1, i^2, i^3, i^4 \\ \mathbf{t}^1, \mathbf{t}^2, \mathbf{t}^3, \mathbf{t}^4}} \sum_{\mathbf{J} \in W_3} \prod_{v=1}^4 (A[i^v, \mathbf{t}^v, \mathbf{j}^v] - EA[i^v, \mathbf{t}^v, \mathbf{j}^v]) \right) \\ &= \Phi_1 + \Phi_2 + \Phi_3. \end{aligned}$$

For given $j_1^1, \dots, j_k^4, t_1^1, \dots, t_k^4, i^1, \dots, i^4$, suppose $E(\prod_{v=1}^4 (A[i^v, \mathbf{t}^v, \mathbf{j}^v] - EA[i^v, \mathbf{t}^v, \mathbf{j}^v])) \neq$

0. Then $\sum_{i=1}^k j_i^v = 0$ ($1 \leq v \leq 4$) and from independence conditions we know that:

$$\forall j_u^v, \exists (u_1, v_1) \neq (u, v) \text{ s.t. } |j_u^v| = |j_{u_1}^{v_1}| \quad (3.5)$$

and that

$$\forall v_1, \exists v_2 \neq v_1 \text{ and } u_1, u_2 \text{ s.t. } |j_{u_1}^{v_1}| = |j_{u_2}^{v_2}|; \quad (3.6)$$

otherwise,

$$\begin{aligned}
& E\left(\prod_{v=1}^4 (A[i^v, \mathbf{t}^v, \mathbf{j}^v] - EA[i^v, \mathbf{t}^v, \mathbf{j}^v])\right) \\
&= E\left((A[i^{v_1}, \mathbf{t}^{v_1}, \mathbf{j}^{v_1}] - EA[i^{v_1}, \mathbf{t}^{v_1}, \mathbf{j}^{v_1}])\right) \cdot E\left(\prod_{\substack{1 \leq v \leq 4 \\ v \neq v_1}} (A[i^v, \mathbf{t}^v, \mathbf{j}^v] - EA[i^v, \mathbf{t}^v, \mathbf{j}^v])\right) \\
&= 0.
\end{aligned}$$

We now evaluate Φ_1 , Φ_2 and Φ_3 .

Evaluation of Φ_1 . Suppose $p \in \mathbb{N}$ and $p \leq 2k - 2$. There are at most $R_{k,p} \cdot (b_N + 1)^p \cdot 2^{4k}$ different \mathbf{J} 's that satisfy $p(\mathbf{J}) = p$, where $R_{k,p} = \#\{\pi = \{U_1, \dots, U_p\} \mid \pi \in \mathcal{P}(4k)\}$. So we have

$$\begin{aligned}
\Phi_1 &\leq \sum_{p=1}^{2k-2} \frac{1}{N^{2k+4} m^{2k}} \cdot N^4 \cdot m^{4k} \cdot R_{k,p} \cdot (b_N + 1)^p \cdot 2^{4k} \cdot M \\
&\leq C \cdot b_N^{-2}
\end{aligned}$$

where

$$M = \sup_N \left\{ \max_{\substack{1 \leq i^v \leq N \\ \mathbf{J} \in \{-b_N, \dots, b_N\}^{4k} \\ \mathbf{t}^v \in \{1, \dots, m\}^k}} \left| E\left[\prod_{v=1}^4 (A[i^v, \mathbf{t}^v, \mathbf{j}^v] - EA[i^v, \mathbf{t}^v, \mathbf{j}^v])\right] \right| \right\}$$

and C is independent of N . From (1.1), (1.2) and (1.3), we know that such M exists.

Evaluation of Φ_2 . Suppose $\mathbf{J} \in W_2$. For any $v \in \{1, 2, 3, 4\}$, if $\exists u \in \{1, 2, \dots, k\}$ such that $\forall u_1 \neq u$, $|j_u^v| \neq |j_{u_1}^v|$, then j_u^v and its absolute value are determined by $j_1^v, \dots, j_{u-1}^v, j_{u+1}^v, \dots, j_k^v$ for $\sum_{i=1}^k j_i^v = 0$. Thus

$$\begin{aligned}
& \#\left\{ \mathbf{J} = (j_1^1, \dots, j_k^4) \in W_2 \mid \forall v, \sum_{l=1}^k j_l^v = 0 \text{ and } \exists j_{u_1}^{v_1} \text{ s.t. } \forall u_2 \neq u_1, |j_{u_1}^{v_1}| \neq |j_{u_2}^{v_1}| \right\} \\
&\leq Q_k \cdot (b_N + 1)^{2k-2} \cdot 2^{4k}. \tag{3.7}
\end{aligned}$$

where $Q_k = \#\{\pi = \{V_1, \dots, V_{2k-1}\} \mid \pi \in \mathcal{P}(4k)\}$. When $\mathbf{J} \in W_2$, there are two situations. One is that $\exists i_1, i_2$ such that $\#S_{i_1} = \#S_{i_2} = 3$ and for any

$i \notin \{i_1, i_2\}$, $\#S_i = 2$. The other situation is that $\exists i_1$ such that $\#S_{i_1} = 4$ and for any $i \neq i_1$, $\#S_i = 2$. In the first situation, suppose $S_{i_1} = \{j_x^a, j_y^b, j_z^c\}$, $S_{i_2} = \{j_w^d, j_u^e, j_v^f\}$. If $a = b = c$ and $d = e = f$, then we can find $g \in \{1, 2, 3, 4\} \setminus \{a, d\}$. Then for any r , there is one unique element which has the same absolute value as j_r^g in $S_{\mathbf{J}}$. Then from (3.6) we know that in $\{j_1^g, \dots, j_k^g\}$ there is at least one element which has a different absolute value from the others and then (3.7) is obtained. If $a \neq b$ and $a \neq c$, then j_x^a has a different absolute value from the other elements in $\{j_1^a, \dots, j_k^a\}$ and then (3.7) is also obtained. In the second situation, suppose $S_{i_1} = \{j_x^a, j_y^b, j_z^c, j_w^d\}$. If $\{a, b, c, d\} \neq \{1, 2, 3, 4\}$ then similarly to the first situation we can get (3.7). If $\{a, b, c, d\} = \{1, 2, 3, 4\}$, then j_x^a has different absolute value from the other elements in $\{j_1^a, \dots, j_k^a\}$ thus we also get (3.7). From the above discussion we know that for any $\mathbf{J} \in W_2$ such that $E(\prod_{v=1}^4 (A[i^v, \mathbf{t}^v, \mathbf{j}^v] - EA[i^v, \mathbf{t}^v, \mathbf{j}^v])) \neq 0$, (3.7) is satisfied. Then we get

$$\begin{aligned}
& \Phi_2 \\
&= \frac{1}{N^{2k+4}m^{2k}} E\left(\sum_{\substack{i^1, i^2, i^3, i^4 \\ \mathbf{t}^1, \mathbf{t}^2, \mathbf{t}^3, \mathbf{t}^4}} \sum_{\mathbf{J} \in W_2} E\left(\prod_{v=1}^4 (A[i^v, \mathbf{t}^v, \mathbf{j}^v] - EA[i^v, \mathbf{t}^v, \mathbf{j}^v])\right)\right) \\
&\leq \frac{1}{N^{2k+4}m^{2k}} N^4 m^{4k} Q_k(b_N + 1)^{2k-2} 2^{4k} M \\
&\leq D \cdot b_N^{-2}
\end{aligned}$$

where D is independent of N .

Evaluation of Φ_3 . When $\mathbf{J} \in W_3$, we have $\#S_i = 2$ for all $i \in \{1, \dots, 2k\}$. Like the situations mentioned above, we have: $\exists u_1 \in \{1, 2, \dots, k\}$ such that $\forall u_2 \neq u_1$, $|j_{u_1}^1| \neq |j_{u_2}^1|$, then $j_{u_1}^1$ and its absolute value are determined by $j_1^1, \dots, j_{u_1-1}^1, j_{u_1+1}^1, \dots, j_k^1$ for $\sum_{i=1}^k j_i^1 = 0$. Suppose $|j_{u_3}^e| = |j_{u_1}^1|$ and $f \in \{1, 2, 3, 4\} \setminus \{1, e\}$, then $\exists u_4 \in \{1, 2, \dots, k\}$ such that $\forall u_5 \neq u_4$, $|j_{u_5}^f| \neq |j_{u_4}^f|$, then $j_{u_4}^f$ and its absolute value are determined by $j_1^f, \dots, j_{u_4-1}^f, j_{u_4+1}^f, \dots, j_k^f$ for $\sum_{i=1}^k j_i^f = 0$. For $j_{u_1}^1$ and $j_{u_4}^f$ are determined, we

have

$$\begin{aligned} & \# \left\{ \mathbf{J} = (j_1^1, \dots, j_k^1, \dots, j_k^4) \in W_3 \mid E \left(\prod_{v=1}^4 (A[i^v, \mathbf{t}^v, \mathbf{j}^v] - EA[i^v, \mathbf{t}^v, \mathbf{j}^v]) \right) \neq 0 \right\} \\ & \leq (b_N + 1)^{p-2} \cdot 2^{4k} \cdot \# \mathcal{P}_2(4k) \end{aligned}$$

and thus

$$\begin{aligned} & \Phi_3 \\ &= \frac{1}{N^{2k+4} m^{2k}} E \left(\sum_{\substack{i^1, i^2, i^3, i^4 \\ \mathbf{t}^1, \mathbf{t}^2, \mathbf{t}^3, \mathbf{t}^4}} \sum_{\mathbf{J} \in W_3} E \left(\prod_{v=1}^4 (A[i^v, \mathbf{t}^v, \mathbf{j}^v] - EA[i^v, \mathbf{t}^v, \mathbf{j}^v]) \right) \right) \\ &\leq \frac{1}{N^{2k+4} m^{2k}} N^4 m^{4k} (b_N + 1)^{p-2} \cdot 2^{4k} \cdot \# \mathcal{P}_2(4k) M \\ &\leq F \cdot b_N^{-2} \end{aligned}$$

where F is independent of N . Finally,

$$\frac{1}{N^4} E((\text{tr} X_N^k - E(\text{tr} X_N^k))^4) = \Phi_1 + \Phi_2 + \Phi_3 \leq (C + D + F) \cdot b_N^{-2}.$$

For $b_N = N - 1$, (3.4) is proved and we have proved the theorem. \square

Remark 3.3 (Hankel block matrices and complex Toeplitz case). If $H_N = (A_{N+1-i-j})$ is a block Hankel matrix which is defined as in the introduction, then the eigenvalue distribution of H_N/\sqrt{mN} converges almost surely to a distribution $\gamma_H^{(m)}$ which is determined by its even moments

$$m_{2k}(\gamma_H^{(m)}) = \sum_{\pi \in \mathcal{P}_2^1(2k)} \frac{r(m, \pi)}{m^{k+1}} \int_{[0,1] \times [-1,1]^k} \prod_{j=1}^{2k} I_{[0,1]}(x_0 - \sum_{q=1}^j (-1)^q x_{\pi(q)}) \prod_{l=0}^k dx_l \quad (3.8)$$

where $r(m, \pi) = \# \{ (t_1, \dots, t_{2k}) \mid E(a_{t_1 t_2}(j_1), \dots, a_{t_{2k} t_1}(j_{2k})) \neq 0 \}$.

Suppose $\pi = \{ \{a_1, b_1\}, \dots, \{a_k, b_k\} \}$, then

$$\begin{aligned} & r(m, \pi) \\ &= \# \{ (t_1, \dots, t_{2k}) \mid 1 \leq t_{a_i}, t_{b_i} \leq m; \left\{ \begin{array}{l} t_{a_i} = t_{b_i} \\ t_{a_{i+1}} = t_{b_{i+1}} \end{array} \right\} \text{ or } \left\{ \begin{array}{l} t_{a_i} = t_{b_{i+1}} \\ t_{a_{i+1}} = t_{b_i} \end{array} \right\} \quad (1 \leq i \leq k) \} \\ &= \# \{ (t_1, \dots, t_{2k}) \mid 1 \leq t_{a_i}, t_{b_i} \leq m; t_{a_i} = t_{(b_i+1)}; t_{(a_i+1)} = t_{b_i} \quad (1 \leq i \leq k) \} \\ &+ O(m^k) \\ &= m^{2k-f(\pi)} + O(m^k) \end{aligned} \quad (3.9)$$

If the blocks of H_N are Hermitian matrices, the results would be similar. If the blocks in a block Toeplitz matrix $T = (A_{i-j})_{i,j=1}^N$ are complex matrices and $A_{-s} = (\overline{A_s})^T$, then the results would also be similar.

Remark 3.4 (relation with dynamical system). $f(\pi)$ has a relation with dynamical system. Let $\pi = \{V_1, V_2, \dots, V_k\}$, $V_r = \{a_r, b_r\}$ ($1 \leq r \leq k$), then $f(\pi)$ is the number of linearly independent equations of (3.1). Now we consider a discrete dynamical system. Give $[2k] = \{1, \dots, 2k\}$ the discrete topology. Let ϕ be a self-homeomorphism on $[2k]$ and $\phi(a_i) = b_i + 1$, $\phi(b_i) = a_i + 1$ ($1 \leq i \leq k$). Consider a continuous map $\psi : \mathbb{Z} \times [2k] \rightarrow [2k]$ such that $\psi(s, t) = \phi^s(t)$ ($s \in \mathbb{Z}$, $t \in [2k]$) then ψ becomes a dynamical system. Obviously we see that the number of orbits of this dynamical system equals the number of independent variables in (3.1) and then equals $2k - f(\pi)$.

4 Block Toeplitz and Hankel Band Matrices

Definition 4.1. Let $T_N = (A_{i-j})_{i,j=1}^N$ be an $mN \times mN$ block Toeplitz matrix. We call T a block Toeplitz band matrix if $\exists b_N < N$ s.t. $A_s = 0$ when $|s| > b_N$. We call b_N the bandwidth of the matrix.

Theorem 4.2 (proportional growth). *Let T_N be a block Toeplitz band matrix with the bandwidth $b_N \sim bN$, $b \in (0, 1]$. Take the normalization*

$$X_N = T_N / \sqrt{m(2-b)bN}.$$

With the notation and assumptions of Theorem 3.2, μ_{X_N} converges almost surely to a symmetric probability distribution $\gamma_T^{(m)}(b)$ which is determined by its even moments

$$m_{2k}(\gamma_T^{(m)}(b)) = \frac{1}{(2-b)^k} \sum_{\pi \in \mathcal{P}_2(2k)} m^{k-1-f(\pi)} \int_{[0,1] \times [-1,1]^k} \prod_{j=1}^{2k} I_{[0,1]}(x_0 + b \sum_{i=1}^j \epsilon_\pi(i) x_{\pi(i)}) \prod_{l=0}^k dx_l$$

where $\epsilon_\pi(i) = 1$ if i is the smaller number of $V_{\pi(i)}$ and $\epsilon_\pi(i) = -1$ otherwise.

Theorem 4.3 (slow growth). *Let T_N be a block Toeplitz band matrices with the bandwidth $b_N = o(N)$ but $b_N \rightarrow \infty$. Take the normalization*

$$X_N = T_N / (\sqrt{2mb_N}).$$

With the notation and assumptions of Theorem 3.2, μ_{X_N} converges weakly to a distribution $\gamma^{(m)}$ which is determined by its even moments

$$m_{2k}(\gamma^{(m)}) = \sum_{\pi \in \mathcal{P}_2(2k)} m^{k-1-f(\pi)}.$$

In addition, if there exist positive constants ϵ_0 and C such that

$$b_N \geq C \cdot N^{\frac{1}{2}+\epsilon_0},$$

then μ_{X_N} converges almost surely to $\gamma^{(m)}$.

Proof of Theorem 4.2. It is easy to see that Lemma 2.2 is also right for block Toeplitz band matrices but now b_N is no longer $N-1$. Let

$$m_{k,N} = \int x^k d\mu_{X_N} = \frac{1}{mN} E(\text{tr} X_N^k) = \frac{1}{mN} (m(2-b)bN)^{-\frac{k}{2}} E(\text{tr} T_N^k).$$

Using Lemma 2.2, we get

$$\begin{aligned} m_{k,N} &= \frac{1}{(2-b)^{\frac{k}{2}}} \frac{1}{mN} \sum_{i=1}^N \sum_{j_1, \dots, j_k = -b_N}^{b_N} \sum_{t_1, \dots, t_k = 1}^m \frac{E(a_{t_1 t_2}(j_1) \cdots a_{t_k t_1}(j_k))}{(mbN)^{\frac{k}{2}}} \prod_{l=1}^k I_{[1,N]}(i + \sum_{q=1}^l j_q) \delta_{0, \sum_{q=1}^k j_q} \\ &= \frac{1}{(2-b)^{\frac{k}{2}}} \left(\frac{1}{b}\right)^{\frac{k}{2}} \sum_{i=1}^N \sum_{j_1, \dots, j_k = -b_N}^{b_N} \sum_{t_1, \dots, t_k = 1}^m \frac{E(a_{t_1 t_2}(j_1) \cdots a_{t_k t_1}(j_k))}{(mN)^{\frac{k}{2}+1}} \prod_{l=1}^k I_{[1,N]}(i + \sum_{q=1}^l j_q) \delta_{0, \sum_{q=1}^k j_q}. \end{aligned}$$

Similarly as in the proof of Theorem 3.2, we know that when k is odd,

$$\sum_{i=1}^N \sum_{j_1, \dots, j_k = -b_N}^{b_N} \sum_{t_1, \dots, t_k = 1}^m \frac{E(a_{t_1 t_2}(j_1) \cdots a_{t_k t_1}(j_k))}{(mN)^{\frac{k}{2}+1}} \prod_{l=1}^k I_{[1,N]}(i + \sum_{q=1}^l j_q) \delta_{0, \sum_{q=1}^k j_q} = o(1),$$

and then $m_{k,N} = o(1)$.

For $m_{2k,N}$,

$$\begin{aligned} &\sum_{i=1}^N \sum_{j_1, \dots, j_{2k} = -b_N}^{b_N} \sum_{t_1, \dots, t_{2k} = 1}^m \frac{E(a_{t_1 t_2}(j_1) \cdots a_{t_{2k} t_1}(j_{2k}))}{(mN)^{k+1}} \prod_{l=1}^{2k} I_{[1,N]}(i + \sum_{q=1}^l j_q) \delta_{0, \sum_{q=1}^{2k} j_q} \\ &= N^{-k-1} \sum_{i=1}^N \sum_{\pi \in \mathcal{P}_2(2k)} \sum_{x_1, \dots, x_k = -b_N}^{b_N} m^{k-1-f(\pi)} \prod_{l=1}^{2k} I_{[1,N]}(i + \sum_{q=1}^l \epsilon_\pi(q) x_{\pi(q)}) + o(1). \end{aligned}$$

So

$$m_{2k,N} = \frac{1}{(2-b)^k} \left(\frac{1}{b}\right)^k N^{-k-1} \sum_{i=1}^N \sum_{\pi \in \mathcal{P}_2(2k)} \sum_{x_1, \dots, x_k = -b_N}^{b_N} m^{k-1-f(\pi)} \prod_{l=1}^{2k} I_{[1,N]}(i + \sum_{q=1}^l \epsilon_\pi(q) x_{\pi(q)}) + o(1),$$

thus

$$\lim_{N \rightarrow \infty} m_{2k,N} = \frac{1}{(2-b)^k} \sum_{\pi \in \mathcal{P}_2(2k)} m^{k-1-f(\pi)} \int_{[0,1] \times [-1,1]^k} \prod_{j=1}^{2k} I_{[0,1]}(x_0 + b \sum_{q=1}^j \epsilon_\pi(q) x_{\pi(q)}) \prod_{l=0}^k dx_l.$$

Let $m_{2k} = \lim_{N \rightarrow \infty} m_{2k,N}$ and $m_{2k-1} = 0$ ($k \in \mathbb{N}$), then for any $k \in \mathbb{N}$ we have

$\lim_{N \rightarrow \infty} m_{k,N} = m_k$. It is easy to see that

$$m_{2k} \leq \sum_{\pi \in \mathcal{P}_2(2k)} m^{k-1-f(\pi)} \leq (2k-1)!! \cdot m^{k-1}$$

and then using Carleman's theorem (see [6]) we know that the limit distribution $\gamma_T^{(m)}(b)$ is determined by its even moments $\{m_{2k}\}$. Similar to the proof of Theorem 3.2 we can easily prove that μ_{X_N} converges almost surely to $\gamma_T^{(m)}(b)$. Then Theorem 4.2 is proved. \square

Proof of Theorem 4.3. Let

$$\begin{aligned} m_{k,N} &= \int x^k d\mu_{X_N} = \frac{1}{mN} E(\text{tr} X_N^k) = \frac{1}{mN} (2mb_N)^{-\frac{k}{2}} E(\text{tr} T_N^k) \\ &= \frac{(mN)^{\frac{k}{2}+1}}{mN(2mb_N)^{\frac{k}{2}}} \sum_{i=1}^N \sum_{j_1, \dots, j_k = -b_N}^{b_N} \sum_{t_1, \dots, t_k = 1}^m \frac{E(a_{t_1 t_2}(j_1) \cdots a_{t_k t_1}(j_k))}{(mN)^{\frac{k}{2}+1}} \prod_{l=1}^k I_{[1,N]}(i + \sum_{q=1}^l j_q) \delta_{0, \sum_{q=1}^k j_q}. \end{aligned}$$

Similarly as in the proof of Theorem 3.2, we know that when k is odd, $m_{k,N} = o(1)$. For $m_{2k,N}$,

$$\begin{aligned} &\sum_{i=1}^N \sum_{j_1, \dots, j_{2k} = -b_N}^{b_N} \sum_{t_1, \dots, t_{2k} = 1}^m \frac{E(a_{t_1 t_2}(j_1) \cdots a_{t_{2k} t_1}(j_{2k}))}{(mN)^{k+1}} \prod_{l=1}^{2k} I_{[1,N]}(i + \sum_{q=1}^l j_q) \delta_{0, \sum_{q=1}^{2k} j_q} \\ &= N^{-k-1} \sum_{i=1}^N \sum_{\pi \in \mathcal{P}_2(2k)} \sum_{x_1, \dots, x_k = -b_N}^{b_N} m^{k-1-f(\pi)} \prod_{l=1}^{2k} I_{[1,N]}(i + \sum_{q=1}^l \epsilon_\pi(q) x_{\pi(q)}) + o(1). \end{aligned}$$

So

$$\begin{aligned}
& m_{k,N} \\
&= \frac{m^{k+1}}{mN(2mb_N)^k} \sum_{i=1}^N \sum_{\pi \in \mathcal{P}_2(2k)} \sum_{x_1, \dots, x_k = -b_N}^{b_N} m^{k-1-f(\pi)} \prod_{l=1}^{2k} I_{[1,N]}(i + \sum_{q=1}^l \epsilon_\pi(q) x_{\pi(q)}) + o(1) \\
&= \frac{1}{N \cdot 2^k \cdot b_N^k} \sum_{i=1}^N \sum_{\pi \in \mathcal{P}_2(2k)} \sum_{x_1, \dots, x_k = -b_N}^{b_N} m^{k-1-f(\pi)} \prod_{l=1}^{2k} I_{[1,N]}(i + \sum_{q=1}^l \epsilon_\pi(q) x_{\pi(q)}) + o(1).
\end{aligned}$$

Thus

$$\begin{aligned}
\lim_{N \rightarrow \infty} m_{2k,N} &= \frac{1}{2^k} \sum_{\pi \in \mathcal{P}_2(2k)} m^{k-1-f(\pi)} \int_{[0,1] \times [-1,1]^k} \prod_{l=0}^k dx_l \\
&= \sum_{\pi \in \mathcal{P}_2(2k)} m^{k-1-f(\pi)}.
\end{aligned}$$

Let $m_{2k} = \lim_{N \rightarrow \infty} m_{2k,N}$ and $m_{2k-1} = 0 (k \in \mathbb{N})$, then for any $k \in \mathbb{N}$ we have $\lim_{N \rightarrow \infty} m_{k,N} = m_k$. It is easy to see that

$$m_{2k} \leq \sum_{\pi \in \mathcal{P}_2(2k)} m^{k-1-f(\pi)} \leq (2k-1)!! \cdot m^{k-1}$$

and then using Carleman's theorem (see [6]) we know that μ_{X_N} converges weakly to a distribution $\gamma^{(m)}$ which is determined by its even moments $\{m_{2k}\}$. Similarly as in the the proof of Theorem 3.2 we know that

$$\frac{1}{N^4} E\left((\text{tr} X_N^k - E(\text{tr} X_N^k))^4\right) \leq B \cdot b_N^{-2}$$

where B is a constant and is independent of N . For $b_N \geq C \cdot N^{\frac{1}{2} + \epsilon_0}$, we have

$$\frac{1}{N^4} E\left((\text{tr} X_N^k - E(\text{tr} X_N^k))^4\right) \leq B \cdot C^{-2} \cdot N^{-1-2\epsilon_0}.$$

Therefore

$$\sum_{N=1}^{\infty} \frac{1}{N^4} E\left((\text{tr} X_N^k - E(\text{tr} X_N^k))^4\right) < \infty$$

and then μ_{X_N} converges almost surely to the limit distribution $\gamma^{(m)}$. \square

Remark 4.4. In [15], the authors proved that the limit of eigenvalue distribution for band Toeplitz random matrix with bandwidth $b_N = o(N)$ is the standard normal distribution $N(0,1)$. But in this paper, the matrix is block Toeplitz and the conclusion will be different. The expectation of $\gamma^{(m)}$ is 0 and the variance of $\gamma^{(m)}$ is $\sum_{\pi \in \mathcal{P}_2(2)} m^{-f(\pi)} = 1$. The forth moment of $\gamma^{(m)}$ is

$\sum_{\pi \in \mathcal{P}_2(4)} m^{1-f(\pi)} = 2 + \frac{1}{m^2}$ which is not the forth moment of $N(0,1)$ if $m \neq 1$. So $\gamma^{(m)}$ is not $N(0,1)$ when $m \neq 1$.

5 Convergence to Semicircle Law

Suppose $\pi = \{\{a_1, b_1\}, \dots, \{a_k, b_k\}\} \in \mathcal{P}_2(2k)$, the system of linear equations determined by π is (see Definition 3.1)

$$\begin{cases} t_{a_i} = t_{b_i+1} & 1 \leq i \leq k \\ t_{b_i} = t_{a_i+1} & 1 \leq i \leq k \end{cases}$$

where $t_{2k+1} = t_1$. This system can be rewritten as

$$\begin{cases} t_{s_1(1)} = t_{s_1(2)} = \dots = t_{s_1(r_1)} = t_{s_1(1)} \\ \dots \\ t_{s_p(1)} = t_{s_p(2)} = \dots = t_{s_p(r_p)} = t_{s_p(1)} \end{cases}$$

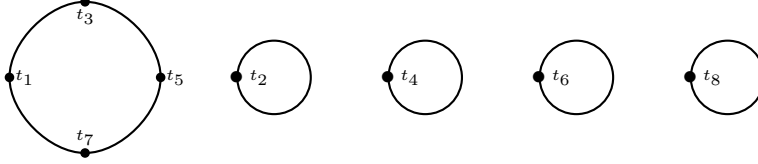
such that $\{s_1(1), \dots, s_1(r_1)\} \cup \dots \cup \{s_p(1), \dots, s_p(r_p)\} = \{1, 2, \dots, 2k\}$ and $\{s_i(1), \dots, s_i(r_i)\} \cap \{s_j(1), \dots, s_j(r_j)\} = \emptyset$ if $i \neq j$. We call $t_{s_i(1)} = t_{s_i(2)} = \dots = t_{s_i(r_i)} = t_{s_i(1)}$ a circle of π . We use $g(\pi)$ to denote the number of the circles of π . For example, if $\pi = \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}\} \in \mathcal{P}_2(8)$, then the system of linear equations determined by π is

$$\begin{cases} t_1 = t_3 \\ t_2 = t_2 \\ t_3 = t_5 \\ t_4 = t_4 \\ t_5 = t_7 \\ t_6 = t_6 \\ t_7 = t_1 \\ t_8 = t_8 \end{cases}.$$

This system can be rewritten as

$$\begin{cases} t_1 = t_3 = t_5 = t_7 = t_1 \\ t_2 = t_2 \\ t_4 = t_4 \\ t_6 = t_6 \\ t_8 = t_8 \end{cases}.$$

So this π has five circles: $t_1 = t_3 = t_5 = t_7 = t_1$, $t_2 = t_2$, $t_4 = t_4$, $t_6 = t_6$ and $t_8 = t_8$, thus $g(\pi) = 5$. These circles can be denoted by



For $f(\pi)$ denotes the number of independent equations of the system, it is easy to see that $g(\pi) = 2k - f(\pi)$ for any $\pi \in \mathcal{P}_2(2k)$.

Lemma 5.1. *For any $\pi \in \mathcal{P}_2(2k)$, $g(\pi) \leq k + 1$.*

The proof of Lemma 5.1 can be found in [22].

We now review the concept of noncrossing partition (see [21]). A partition $\pi \in \mathcal{P}(n)$ is called noncrossing if whenever four elements $1 \leq a < b < c < d \leq n$ are such that $a \sim_\pi c$ and $b \sim_\pi d$, then $a \sim_\pi b \sim_\pi c \sim_\pi d$.

Lemma 5.2. *For any $\pi \in \mathcal{P}_2(2k)$, $g(\pi) = k + 1$ if and only if π is noncrossing.*

The proof of Lemma 5.2 can be found in [20].

Theorem 5.3. *Suppose $\gamma_T^{(m)}$ is defined as in Theorem 3.2. As $m \rightarrow \infty$, $\gamma_T^{(m)}$ converges weakly to the semicircle law $w(x)$, i.e.,*

$$w(x) = \begin{cases} \frac{1}{2\pi} \sqrt{4 - x^2} & |x| \leq 2, \\ 0 & |x| > 2. \end{cases}$$

Proof of Theorem 5.3. From Theorem 3.2 we know that the odd moments of $\gamma_T^{(m)}$ are all zero and its even moments are

$$m_{2k}(\gamma_T^{(m)}) = \sum_{\pi \in \mathcal{P}_2(2k)} m^{k-1-f(\pi)} \int_{[0,1] \times [-1,1]^k} \prod_{j=1}^{2k} I_{[0,1]}(x_0 + \sum_{q=1}^j \epsilon_\pi(q) x_{\pi(q)}) \prod_{l=0}^k dx_l$$

where

$$\epsilon_\pi(q) = \begin{cases} 1 & \text{if } q \text{ is the smaller element of } V_{\pi(q)}, \\ -1 & \text{if } q \text{ is the larger element of } V_{\pi(q)}. \end{cases}$$

From Lemma 5.1 and Lemma 5.2 we know that if $\pi \in \mathcal{P}_2(2k)$, then $f(\pi) = k - 1$ if π is noncrossing, otherwise $f(\pi) > k - 1$. So we have

$$\lim_{m \rightarrow \infty} m_{2k}(\gamma_T^{(m)}) = \sum_{\substack{\pi \in \mathcal{P}_2(2k) \\ \pi \text{ is noncrossing}}} \int_{[0,1] \times [-1,1]^k} \prod_{j=1}^{2k} I_{[0,1]}(x_0 + \sum_{q=1}^j \epsilon_\pi(q) x_{\pi(q)}) \prod_{l=0}^k dx_l.$$

From [4] we know that when π is noncrossing,

$$\int_{[0,1] \times [-1,1]^k} \prod_{j=1}^{2k} I_{[0,1]}(x_0 + \sum_{q=1}^j \epsilon_\pi(q) x_{\pi(q)}) \prod_{l=0}^k dx_l = 1.$$

So

$$\lim_{m \rightarrow \infty} m_{2k}(\gamma_T^{(m)}) = \sum_{\substack{\pi \in \mathcal{P}_2(2k) \\ \pi \text{ is noncrossing}}} 1.$$

From [11] we know that

$$\#\{\pi \in \mathcal{P}_2(2k) \mid \pi \text{ is noncrossing}\} = C_k$$

where C_k is Catalan number. So

$$\lim_{m \rightarrow \infty} m_{2k}(\gamma_T^{(m)}) = C_k.$$

For C_k is the $2k$ -th moment of $w(x)$ whose odd moments are all zero, we know that $\gamma_T^{(m)}$ converges weakly to $w(x)$ thus the theorem is proved. \square

Remark 5.4. If the assumptions in Theorem 4.3 are all satisfied, then it is easy to see that as $m \rightarrow \infty$, $\gamma^{(m)}$ converges weakly to the semicircle law $w(x)$.

Remark 5.5. For a block Hankel matrix as we discussed in Remark 3.3 we also have that as $m \rightarrow \infty$, $\gamma_H^{(m)}$ converges weakly to the semicircle law $w(x)$ because (see (3.9))

$$\begin{aligned} r(m, \pi) &= \#\{(t_1, \dots, t_{2k}) \mid 1 \leq t_{a_i}, t_{b_i} \leq m; t_{a_i} = t_{(b_i+1)}; t_{(a_i+1)} = t_{b_i} (1 \leq i \leq k)\} \\ &+ O(m^k) \\ &= m^{2k-f(\pi)} + O(m^k) \end{aligned}$$

and $\{\pi \in \mathcal{P}_2(2k) \mid \pi \text{ is noncrossing}\} \subset \mathcal{P}_2^1(2k)$ and the fact that the integral in (3.8) is 1 when π is noncrossing (see [4]). If the blocks of H_N are Hermitian matrices, the results would be similar. If the blocks in a block Toeplitz matrix $T = (A_{i-j})_{i,j=1}^N$ are complex matrices and $A_{-s} = (\overline{A_s})^T$, then the results would also be similar.

Remark 5.6. If the blocks in a block Toeplitz matrix $T = (A_{i-j})_{i,j=1}^N$ are symmetric matrices and $A_{-s} = A_s$, just like those discussed in [18], then the eigenvalue distribution of T/\sqrt{mN} converges almost surely to a distribution $\widetilde{\gamma}_T^{(m)}$ which is determined by its even moments

$$m_{2k}(\widetilde{\gamma}_T^{(m)}) = \sum_{\pi \in \mathcal{P}_2(2k)} \frac{r(m, \pi)}{m^{k+1}} \int_{[0,1] \times [-1,1]^k} \prod_{j=1}^{2k} I_{[0,1]}(x_0 + \sum_{q=1}^j \epsilon_\pi(q) x_{\pi(q)}) \prod_{l=0}^k dx_l$$

where $r(m, \pi)$ is the same as (3.9). So we also have that as $m \rightarrow \infty$, $\widetilde{\gamma}_T^{(m)}$ converges weakly to the semicircle law $w(x)$.

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